WHAT A PAIR!

Nonagenarian mathematician Joan Birman thought her major discoveries were behind her. Then came an email from a teen. **Gregory Barber** tells the story of their collaboration

The note was of a kind Joan Birman had received many times: a plea from a high school student asking for help learning mathematics.

Messages like this had arrived with regularity during Birman's career. They kept arriving via email even after she retired from Columbia University in 2004. Now, in the summer of 2019, she was 92 years old but she wanted to keep doing maths.

And here was this email.

Birman replied she was willing to meet. She told the teenager to find I.N. Herstein's *Topics in Algebra*, a standard torture device for aspiring graduate students.

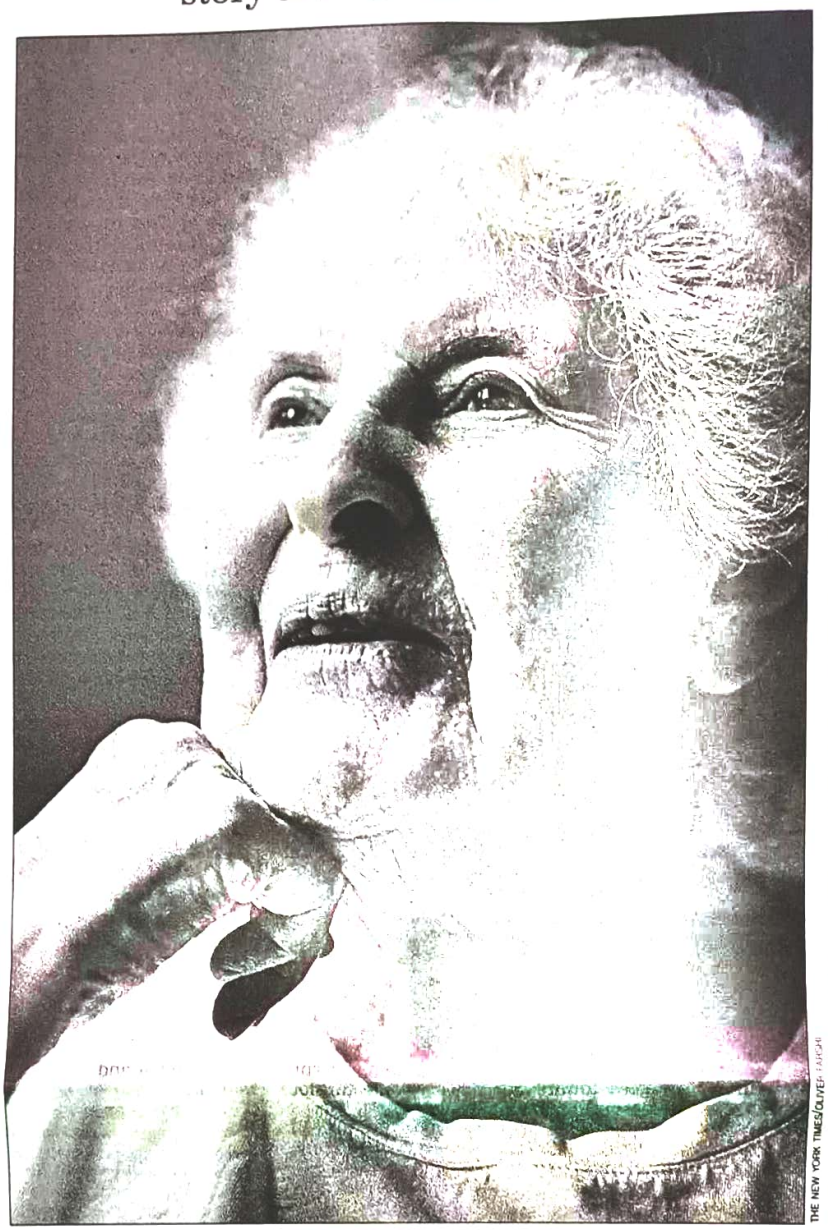
The girl arrived that July in the lobby of Birman's apartment on Riverside Drive. Her name was Vasudha Bharathram, and she had just turned 15 that month.

Birman asked Vasudha what she liked about high school geometry. Her own subfield of mathematics was called topology, Birman explained; it too involved shapes, but ones that were fluid and stretchy, not pinned down by rigid coordinates. Birman took out a pencil and drew an example, called a braid, which twirls around just like the hairstyle. She sent the student off with an assignment to read a chapter of the textbook.

She was surprised when, a few days later, Vasudha sent her another email, this time with photographs of her notebook. There were pages and pages of solutions. Birman was startled by the girl's intrepidness; problems from Herstein were no one's idea of a good time. Then she looked more closely and realised that Vasudha had solved the problems by sketching out unusual paths to the right answer. At their next meeting, she asked her to explain her thinking. There was something original here, Birman realised.

Birman's first move to New York had been at age 12. At her all-girls public high school, she discovered a love of geometry, maths that she could actually see. She worked part-time as an engineer while raising three children. At 34, she started taking evening maths courses at New York University. She passed the doctoral qualifying exam without realising what the test was for.

One interest was knots, which are defined as single strands that loop and twist upon themselves in ways that cannot be undone. A major question in knot theory is telling whether two different-looking knots are actually the same. To decipher their true nature, mathematicians translate shapes into algebra. Expressions called polynomials help identify knots as unique.



THE NEW YORK TIMES/OLIVER JARAK



MATHMAGIC: Joan Birman and Vasudha Bharathram (left) continue to work out complex problems using polynomials along with third collaborator Tara Brendle, who teaches at the University of Glasgow in the UK

Braids, meanwhile, are defined as multiple overlapping strands that don't form loops, instead running off the page. What makes them special is that you can do calculations with them directly — algebra with pictures.

Birman completed her doctorate in 1968. She joined the faculty of Barnard College and quickly became known for connecting the powerful algebra of braids to other topics in mathematics, from knots to chaos. In 1984, when mathematician Vaughan Jones uncovered a new polynomial in the course of studying braids, he came to Birman. She quickly helped him see how it could distinguish one knot from another — a holy grail for the field.

Vasudha would email Birman to set up a meeting each time she finished a Herstein chapter. By early autumn, they were done with that book and had moved on to Birman's own book on braids.

Eventually, Birman directed Vasudha to a chapter in which she described a special way to represent braids with large sets of polynomials. Known as the Burau representation, it came with a question: whether information was lost in this translation from shapes to algebra. This is known as "faithfulness". Burau was known to be faithful for braids with three or fewer strands, and unfaithful for braids with five or more. That left four strands. For as long as Birman had been alive, no one had been able to come to any conclusions about whether Burau would be faithful in that case.

Soon after the pandemic began, Birman suggested that Vasudha take another look at the well-established three-strand case. Perhaps if she could find a new way to prove it, they would see a path others had missed.

At Birman's direction, Vasudha began taking college courses, which were newly online. The meetings with her mentor continued over Zoom.

By the time they could meet in person, Vasudha was formulating a new way to prove the faithfulness for three-strand braids using a special type of polynomial. It was an unusual choice because that very polynomial had played a role in proving why Burau was unfaithful for braids of five or more strands.

One day, Birman invited Tara Brendle, one of her final graduate students at Columbia, to take a look at their work. She quickly spotted an error in one of their diagrams — at that moment, Birman knew they had found a collaborator.

Occasionally, Vasudha and Brendle would meet in person for "maths camp" in Birman's apartment. The trio were satisfied with the new three-strand proof and had moved on to four, using the same polynomial to describe the overlapping twists and turns of the strands. The problem was that certain terms in the polynomial would sometimes cancel out one another, implying that information was potentially lost, i.e., unfaithfulness.

Their task was to show that enough of these cancellations never piled up to cause a problem. But showing that this was always the case would involve thousands of hand-drawn diagrams of various braids and intensive calculations. Then, a realisation altered their course. Rather than labouring case by case, they changed their approach to imagine their four-stranded braids as having five strands. In this world of five-stranded possibilities, they could perform a fix that would make all the cancellations disappear from the polynomial. Watching them vanish was, Brendle said, "the most satisfying moment of my mathematical career".

This July, they posted the paper.

NEW YORK TIMES NEWS SERVICE